HORIZONTAL GRADIENT OF POLYNOMIAL FUNCTIONS FOR THE STANDARD ENGEL STRUCTURE ON $\mathbb{R}^4$

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ABSTRACT. We investigate the set V_f of horizontal critical points of a polynomial function f for the standard Engel structure defined by the 1-forms $\omega_3 = dx_3 - x_1 dx_2$, $\omega_4 = dx_4 - x_3 dx_2$, endowed with the sub-Riemannian metric $g_{SR} = dx_1^2 + dx_2^2$. For a generic polynomial, we show that the intersection of any fiber of f and V_f does not contain a horizontal curve. Then we prove that each trajectory of the horizontal gradient of f approaching the set V_f has a limit.

1. INTRODUCTION

An **Engel structure** Δ is a non-integrable distribution of rank 2 on a 4-dimensional manifold which satisfies the following conditions:

$$rank[\Delta, \Delta] = 3,$$
$$rank[\Delta, [\Delta, \Delta]] = 4,$$

where $[.,.]$ denotes the Lie bracket. Engel structures are stable (or generic) in the sense that all C^2-small perturbation of an Engel structure is still an Engel structure. A manifold with an Engel structure is called an **Engel manifold**. In this paper, we will deal with the standard Engel structure on $\mathbb{R}^4$, defined by the 1-forms $\omega_3 = dx_3 - x_1 dx_2$ and $\omega_4 = dx_4 - x_3 dx_2$, which is generated by the following vector fields:

$$\begin{cases} X_1 = \dfrac{\partial}{\partial x_1} \\ X_2 = \dfrac{\partial}{\partial x_2} + x_1 \dfrac{\partial}{\partial x_3} + x_3 \dfrac{\partial}{\partial x_4} \end{cases}$$

We denote by Δ this Engel structure. By a result of Engel [9], at each point of an Engel manifold, there exists a neighborhood with local coordinates x_1, x_2, x_3, x_4 such that the Engel structure is generated by the vector fields X_1, X_2 above. So locally, all Engel structures are isomorphic.

Let us fix $g_{SR} = dx_1^2 + dx_2^2$, a **sub-Riemannian metric** on Δ for which the system $\{X_1, X_2\}$ is orthonormalized. Let $X_3 = \dfrac{\partial}{\partial x_3}, X_4 = \dfrac{\partial}{\partial x_4}$. Then $g = dx_1^2 + dx_2^2 + \omega_3^2 + \omega_4^2$ is the Riemannian metric on $\mathbb{R}^4$ extending g_{SR} and making the system

$\{X_1, X_2, X_3, X_4\}$ orthonormalized. The pair (Δ, g_{SR}) is called a **sub-Riemannian structure** on $\mathbb{R}^4$.

We refer to [1, 3, 6, 10, 12, 16, 24, 25, 27, 28] for further informations about subriemannian geometry and Engel structures.

Let $\mathbb{R}_d[x] := \{f \in \mathbb{R}[x], \deg f \leq d\}$, the set $\mathbb{R}_d[x]$ is furnished a structure of finite dimensional vector space. We may identify a polynomial $f \in \mathbb{R}_d[x]$ with its coefficient vector and identify $\mathbb{R}_d[x]$ with $\mathbb{R}^{dim \mathbb{R}_d[x]}$. So $\mathbb{R}_d[x]$ is endowed with the usual topology of $\mathbb{R}^{\dim \mathbb{R}_d[x]}$. For each $f \in \mathbb{R}_d[x]$, we can associate a vector field $\nabla^h f$, called **horizontal gradient** of f, which is the projection of the Riemannian gradient ∇f on the distribution Δ with respect to the metric g_{SR}. In the coordinate system $\{X_1, X_2, X_3, X_4\}$,

$$\nabla f = \sum_{i=1}^{4} (X_i f) X_i.$$

By definition

$$\nabla^h f = (X_1 f) X_1 + (X_2 f) X_2.$$

For convenience, we will sometimes identify $\nabla^h f$ with the map $x \mapsto (X_1 f(x), X_2 f(x))$. The notion of horizontal gradient appeared in some recent works [2, 22] on the Carnot groups. Denote

$$V_f = \{\nabla^h f = 0\}$$

the set of horizontal critical points of f. Note that if $x \in V_f$, then either x is a riemannian critical point of f, i.e., $\nabla f(x) = 0$, or $\Delta_x \subset T_x f^{-1}(f(x))$.

Definition 1.1. *An almost everywhere differentiable curve $\gamma : (t_1, t_2) \to \mathbb{R}^4$ is called a horizontal curve if its derivative $\dot{\gamma}(t)$ is almost everywhere contained in $\Delta_{\gamma(t)}$.*

Let us firstly recall the Łojasiewicz gradient inequality [4, 7, 18, 19, 20] for Riemannian gradient ∇f. This inequality is an important tool to study the trajectories of ∇f. Assume that f is an analytic function and $\nabla f(x_0) = 0$, then the Łojasiewicz gradient inequality says that in a neighborhood U_0 of x_0,

$$||\nabla f(x)|| \geq C|f(x) - f(x_0)|^{\alpha},$$

for some $0 < \alpha < 1$ and $C > 0$. As a consequence, in [19, 21], Łojasiewicz proved that locally the length of each trajectory $x(t)$ of ∇f is bounded uniformly, hence the limit $\lim_{t \to \infty} x(t)$ exists, i.e., if $x(t_m) \to x_0$ for some sequence $t_m \to \infty$, then $x(t) \to x_0$ as $t \to \infty$.

In [8], we have consider the trajectories of the horizontal gradient for a class of codimension one distribution, called splitting distribution, on $\mathbb{R}^n$. This class contains, in particular, contact structures. We have noticed that the Łojasiewicz gradient inequality does not hold for horizontal gradient and some new phenomena appear, for example, a trajectory of the horizontal gradient may have unbounded length or accumulate to a cycle. This presents a major obstruction to the study of the horizontal gradient's trajectories since the techniques of Łojasiewicz do not apply to the horizontal gradient. It turns out that to overcome this difficulty, we have to study the restriction of the function on its set of horizontal critical points $f|_{V_f}$. The conclusion is that for a generic polynomial, the restriction $f|_{V_f}$ is a Morse function and the behavior of the trajectories of horizontal gradient is similar to those of Riemannian gradient.

Dealing with the standard Engel structure Δ above, in general, the Łojasiewicz gradient inequality does not hold for horizontal gradient. In fact, if $V_f \neq \emptyset$ and $f|_{V_f}$ is a non constant function, then we can not have the Łojasiewicz gradient inequality. Now studying $f|_{V_f}$ is not sufficient to understand the behavior of the trajectories of $\nabla^h f$, and it turns out that the set Γ_f of points in V_f, where V_f is not transverse to the distribution, plays an important role in the study of the trajectories of $\nabla^h f$.

The paper is organized as follow. In the second section, we recall some basic notions and results of semi-algebraic geometry. In the third part, we study the generic properties of the dimension of V_f. On the generic dimension of V_f, we obtain the following.

Theorem 3.1. *Generically, V_f is smooth and of dimension 2 or is empty, i.e., the set*

$$K_d = \{f \in \mathbb{R}_d[x] : V_f \text{ is smooth of dimension } 2 \text{ or is empty}\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

The proof of this result is based on the transversality theorem with parameters. It's quite similar to the proof of Theorem 4.4 in [8].

Set

$$\Gamma_f := \{x \in V_f : T_x V_f \cap \Delta_x \neq \{0\}\}.$$

The following result plays an important role in the study of the trajectories of horizontal gradient.

Theorem 4.1. *Generically, the set Γ_f is a smooth algebraic curve or an empty set, moreover, Γ_f does not have a connected component which is contained in a fiber of f, i.e., the set*

$$M_d = \{f \in \mathbb{R}_d[x] : \quad \Gamma_f \text{ is a smooth algebraic curve or an empty set and does not} \atop \text{have a connected component contained in a fiber of } f$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

In the last part, we show that

Theorem 5.1. *Generically, the trajectories of $\nabla^h f$ in a compact set B have a limit, i.e., the set*

$$N_d = \{f \in \mathbb{R}_d[x] : \text{ the trajectories of } \nabla^h f \text{ have a limit on } \partial B \cup V_f\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

2. Briefs on transversality and semi-algebraicity

We recall some principal definitions and results of transversality and semi-algebraicity. We refer to [11, 13, 15, 4, 5, 7, 17, 18, 19] for more details. Let X, Y, Z, P be some C^∞ manifolds. Let S be a C^∞ submanifold of Y. Let $f : X \to Y$ be a C^∞ map. Denote $C^\infty(X, Y)$, the set of C^∞ maps from X to Y.

Definition 2.1. *We say that f is transverse to S if $f(X) \cap S = \emptyset$ or for each $x \in f^{-1}(S)$, we have*

$$d_x f(T_x X) + T_{f(x)} S = T_{f(x)} Y.$$

Note that if f is a submersion, then it is transverse to all submanifolds of Y.

Proposition 2.2. [13] *If $f \pitchfork S$, then the inverse image $f^{-1}(S)$ of S is a submanifold of X. Moreover, if $codim_Y S \leq dimX$ and $f(X) \cap S \neq \emptyset$, then*

$$codim_X f^{-1}(S) = codim_Y S.$$

If $codim_Y S > dimX$, then $f \pitchfork S$ if and only if $f(X) \cap S = \emptyset$; in this case, the transversality implies $codim_X f^{-1}(S) < codim_Y S$.

Definition 2.3. *A semi-algebraic set of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ defined by a finite sequence of polynomial equations and inequalities with real coefficients, or any finite union of such sets. In other words, the semi-algebraic subsets of $\mathbb{R}^n$ form the smallest class SA_n of subsets of $\mathbb{R}^n$ such that:*

1. If $P \in \mathbb{R}[x_1, \ldots, x_n]$, then $\{x \in \mathbb{R}^n : P(x) = 0\} \in SA_n$ and $\{x \in \mathbb{R}^n : P(x) > 0\} \in SA_n$.

2. If $A \in SA_n$ and $B \in SA_n$, then $A \cap B$, $A \cup B$ and $\mathbb{R}^n - A$ are in SA_n.

Theorem 2.4. [5] *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. If A is dense in $\mathbb{R}^n$, then there exists a set B which is semi-algebraic open dense in $\mathbb{R}^n$ such that $B \subset A$.*

Let us recall the following theorem on the semi-algebraic local triviality of semi-algebraic mappings.

Theorem 2.5. [14] *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be some semi-algebraic sets. Let $f : A \to B$ be a semi-algebraic map. Then there is a $(\dim B - 1)$-dimensional semi-algebraic subset C of B such that, for each connected component U of $B - C$, the restriction $f|_{f^{-1}(U)} \to U$ is a semi-algebraic fibration.*

We finish this part by recalling the following transversality theorem with parameters.

Theorem 2.6. [11, 13] *(Transversality theorem with parameters) Let $F : P \times X \to Y$ be a C^∞ map. Set $f_p = F(p, .) : X \to Y$. If $F \pitchfork S$, then the set*

$$D = \{p \in P : f_p \pitchfork S\}$$

is open dense in P. Moreover, if X, Y, S, P are semi-algebraic sets and if F is a semi-algebraic map, then D is also semi-algebraic.

Proof. The proof of openness and density of D is done in [11, 13]. The method used also permits to prove that D is semi-algebraic if X, Y, S, P and F are semi-algebraic. □

3. Generic horizontal critical set

In this part, we study the generic dimension of V_f.

Theorem 3.1. *Generically, V_f is smooth and of dimension 2 or is empty, i.e., the set*

$$K_d = \{f \in \mathbb{R}_d[x] : V_f \text{ is smooth of dimension } 2 \text{ or is empty}\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

Proof. Let us write a polynomial $f \in \mathbb{R}_d[x]$ as follows

$$f = \alpha_0 + \sum_{i=1}^{4} \alpha_i x_i + g$$

where g does not have the linear part. Then

$X_1 f = \alpha_1 + X_1 g,$

$X_2 f = \alpha_2 + \alpha_3 x_1 + \alpha_4 x_3 + X_2 g.$

We identify $f \in \mathbb{R}_d[x]$ with its coefficient vector. Set

$$L : \mathbb{R}_d[x] \times \mathbb{R}^4 \quad \rightarrow \quad \mathbb{R}^2$$
$$(f, x) \quad \mapsto \quad ((X_1 f)(x), (X_2 f)(x)).$$

The Jacobian matrix of L contains the following matrix

$$(\frac{\partial L}{\partial \alpha_1}, \frac{\partial L}{\partial \alpha_2}) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

This implies that L is a submersion. Consequently, L is transverse to $\{0\} \subset \mathbb{R}^2$. By Theorem 2.6, the set

$$K'_d := \{f \in \mathbb{R}_d[x] : \nabla^h f = L(f, .) \pitchfork \{0\}\} \tag{1}$$

is semi-algebraic open dense in $\mathbb{R}_d[x]$. For each $f \in K'_d$, by Proposition 2.2 and by this transversality, V_f is smooth and if $V_f \neq \emptyset$, then $codim V_f = codim_{\mathbb{R}^2} \{0\} = 2$, so $dim V_f = 2$. Therefore $K'_d \subset K_d$. This ends the proof of the theorem. $\square$

Proposition 3.2. *Generically, the map $\nabla^h f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is a submersion on V_f, i.e., the set*

$$L_d := \{f \in \mathbb{R}_d[x] : \nabla^h f \text{ is submersive on } V_f\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

Proof. For every $f \in K'_d$, where K'_d is defined by (1), the transversality condition $\nabla^h f \pitchfork \{0\}$ implies that $\nabla^h f$ is a submersion on $(\nabla^h f)^{-1}(\{0\})$, i.e., on V_f. $\square$

If f is an affine form $f = \sum_{i=1}^4 \alpha_i x_i + \alpha_0$, then $X_1 f = \alpha_1$, $X_2 f = \alpha_2 + \alpha_3 x_1 + \alpha_4 x_3$. So if $\alpha_1 \neq 0$, then V_f is empty. Let $K''_1 := \{f \in \mathbb{R}_1[x] : \alpha_1 \neq 0\}$. It is clear that K''_1 is semi-algebraic open dense in $\mathbb{R}_1[x]$ and $K''_1 \subset K_1$. Then we have the following result for the affine case.

Proposition 3.3. *Generically, V_f is empty for $f \in \mathbb{R}_1[x]$, i.e., the set*

$$\underline{K}_1 = \{f \in \mathbb{R}_1[x] : V_f \text{ is empty}\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_1[x]$.

4. GENERICITY OF Γ_f

From now on, we denote $X_{i_1 i_2 \ldots i_k} f := X_{i_1}(X_{i_2}(\ldots(X_{i_k} f)\ldots))$ for $f \in \mathbb{R}_d[x]$. We still write a polynomial $f \in \mathbb{R}_d[x]$ under the following form

$$f = \sum_{i=1}^4 \alpha_i x_i + \frac{1}{2} \sum_{i,j=1}^4 \beta_{ij} x_i x_j + g,$$

where g does not contain monomials of degree $1, 2$. Recall that

$$\Gamma_f := \{x \in V_f : T_x V_f \cap \Delta_x \neq \{0\}\}, \tag{2}$$

the set of points in V_f where V_f is not transverse to the distribution.

The goal of this section is to prove the following result.

Theorem 4.1. *Generically, the set Γ_f is a smooth algebraic curve or an empty set, moreover, Γ_f does not have a connected component which is contained in a fiber of f, i.e., the set*

$$M_d = \{f \in \mathbb{R}_d[x] : \ \Gamma_f \text{ is a smooth algebraic curve or an empty set and does not have a connected component contained in a fiber of } f$$

$$(3)$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

In the case $d = 1$, the proof of Theorem 4.1 is trivial since $V_f = \emptyset$ for all $f = \alpha_0 + \sum_{i=1}^{4} \alpha_i x_i$ with $\alpha_1 \neq 0$. So assume that $d \geq 2$. First of all, we characterize the set of horizontal curves contained in the fibers of f.

The first statement of Theorem 4.1 is proven by the following lemma.

Lemma 4.2. *There exists a semi-algebraic open dense set $B_d \subset \mathbb{R}_d[x]$ such that for every $f \in B_d$, the set Γ_f is a smooth algebraic curve or an empty set.*

Proof. Consider the map

$$\Theta : \mathbb{R}^4 \times \mathbb{R}_d[x] \ \rightarrow \ \mathbb{R}^2 \times M_2$$
$$(x, f) \ \mapsto \ (X_1 f(x), X_2 f(x), X_{11} f(x), X_{21} f(x), X_{22} f(x), X_{12} f(x)),$$

where $M_2 \cong \mathbb{R}^4$ is the space of 2×2 matrices. The Jacobian matrix of Θ contains the following matrix

$$\frac{\partial \Theta}{\partial(\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{22}, \alpha_3)} = \begin{pmatrix} 1 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 1 & 0 & x_1 & x_2 & x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

whose rank is 6. Therefore, Θ is a submersion and hence is transverse to $\{(0,0)\} \times \Sigma_2$ where Σ_2 is the algebraic subset of M_2, constituted of degenerate matrices. Denote by $0I_2$ the zero matrix in M_2. Note that $0I_2$ is the only singular point of Σ_2. By Transversality theorem with parameters 2.6, the sets

$$B_d^1 := \{f \in \mathbb{R}_d[x] : \Theta(., f) \pitchfork \{(0,0)\} \times (\Sigma_2 - \{0I_2\})\},$$

$$B_d^2 := \{f \in \mathbb{R}_d[x] : \Theta(., f) \pitchfork \{(0,0) \times 0I_2\}\}$$

are semi-algebraic open dense in $\mathbb{R}_d[x]$. For every $f \in B_d^2$, by Proposition 2.2, we deduce that $(0,0) \times 0I_2 \notin Im\Theta(., f)$. Let $f \in B_d := B_d^1 \cap B_d^2$. We have $\Theta(., f) \pitchfork \{(0,0)\} \times (\Sigma_2 - \{0I_2\})$ and $Im\Theta(., f)$ does not contain the singular point $\{(0,0) \times 0I_2\}$. Finally, by Proposition 2.2, the set $(\Theta(., f))^{-1}(\{(0,0)\} \times \Sigma_2)$ is smooth and if it is not empty, we have

$$codim_{\mathbb{R}^4}(\Theta(., f))^{-1}(\{(0,0)\} \times \Sigma_2) = codim_{\mathbb{R}^6}\{(0,0)\} \times \Sigma_2 = 3.$$

Hence $dim(\Theta(., f))^{-1}(\{(0,0)\} \times \Sigma_2) = 1$. Note that $x \in (\Theta(., f))^{-1}(\{(0,0)\} \times \Sigma_2)$ if and only if $d_x(\nabla^h f)|_{\Delta_x}$ is degenerate, hence if and only if $T_x V_f \cap \Delta_x \neq \{0\}$. This ends the proof of the lemma.

$$\square$$

Remark 4.3. Is is easy to check that the condition that $d_x(\nabla^h f)|_{\Delta_x}$ is degenerate gives

$$\Gamma_f = \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f X_{22} f - X_{21} f X_{12} f = 0\}. \tag{4}$$

Remark 4.4. If $\deg f \leq d$, then

(i) $\deg X_1 f \leq d - 1$, $\deg X_2 f \leq d$,
(ii) $\deg X_{11} f \leq d - 2$, $\deg X_{21} f \leq d - 1$, $\deg X_{12} f \leq d - 1$, $\deg X_{22} f \leq d$.

Hence by [23, 26], the number of connected components of Γ_f is bounded by $c(d) := 2(d-1)[4(d-1) - 1]^3$.

By classical Morse theory, we have

Lemma 4.5. *There exists a semi-algebraic open dense set $J_d \subset \mathbb{R}_d[x]$ such that for every $f \in J_d$, the set of critical points of f*

$$Cr(f) := \{\nabla f = (X_1 f, X_2 f, X_3 f, X_4 f) = 0\}$$

is finite.

Now we have the following constrain on the intersection $V_f \cap f^{-1}(t)$.

Lemma 4.6. *Let $f \in I_d = B_d \cap J_d$, then for every t, the intersection $V_f \cap f^{-1}(t)$ contains only curves (smooth or not) and points, i.e., $\dim(V_f \cap f^{-1}(t)) \leq 1$.*

Proof. Suppose that $\dim(V_f \cap f^{-1}(t)) > 1$ for some t. Let $S \subset V_f \cap f^{-1}(t)$ be a smooth surface of dimension 2. Since the number of critical points of f is finite, we may assume that S does not contain any critical point of f. By the proof of Lemma 4.2, it follows that $B_d \subset K_d$ where K_d is defined in Theorem 3.1, so $\dim V_f = 2$. For each $x \in S$, we have $T_x V_f = T_x S \subset T_x f^{-1}(f(x))$ and $\Delta_x \subset T_x f^{-1}(f(x))$. Note that $\dim \Delta_x = 2$ and $\dim T_x f^{-1}(f(x)) = 3$, so $T_x V_f \cap \Delta_x \neq \{0\}$. Hence $x \in \Gamma_f$. It follows that $S \subset \Gamma_f$. This contradicts to the fact that Γ_f is a smooth curve for every $f \in B_d$. The lemma is proved. $\square$

If that Γ_f has a connected component which is contained in a fiber of f, then we can calculate the tangent vector of this component as follows.

Proposition 4.7. *Let $f \in I_d = B_d \cap J_d$. Assume that γ is a connected component of Γ_f which is contained in a fiber of f. Then for $x \in \gamma$,*

$$\xi_f(x) = -[X_{21} f(x)] X_1 + [X_{11} f(x)] X_2 \tag{5}$$

is a tangent vector of γ at x.

Proof. Note that by Lemma 4.6 and by assumption, γ is a component of $V_f \cap f^{-1}(t)$ for some t. Now $(d_x(\nabla^h f, f - t))(\xi_f(x)) =$

$$\begin{pmatrix} X_{11} f & X_{21} f & X_{31} f & X_{41} f \\ X_{12} f & X_{22} f & X_{32} f & X_{42} f \\ 0 & 0 & X_3 f & X_4 f \end{pmatrix}_x \cdot \begin{pmatrix} -X_{21} f \\ X_{11} f \\ 0 \\ 0 \end{pmatrix}_x = \begin{pmatrix} 0 \\ X_{11} f X_{22} f - X_{12} f X_{21} f \\ 0 \end{pmatrix}_x .$$

Then $(d_x(\nabla^h f, f - t))(\xi_f(x)) = 0$ by Remark 4.3, for all $x \in \gamma$. This implies that $\xi_f(x)$ is a tangent vector of $V_f \cap f^{-1}(t)$ at x. Hence $\xi_f(x)$ is a tangent vector of γ at x. $\square$

The vector $\xi_f(x)$ does not give a tangent direction of a connected component of Γ_f which is contained in a fiber of f if $\xi_f(x) = 0$. So we are going to show that for a generic polynomial, $\xi_f(x) \neq 0$ on Γ_f excepted at finitely many points.

Recall that $\Gamma_f = \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f X_{22} f - X_{21} f X_{12} f\}$. Let

$$
\begin{aligned}
\Omega_f^1 &:= \{x \in \Gamma_f : X_{11} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f X_{22} f - X_{21} f X_{12} f = X_{11} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f = X_{21} f X_{12} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f = X_{21} f = 0\} \cup \\
&\qquad \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f = X_{12} f = 0\} \\
&=: S_f^1 \cup S_f^2,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_f^2 &:= \{x \in \Gamma_f : X_{21} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{11} f X_{22} f - X_{21} f X_{12} f = X_{21} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{21} f = X_{11} f X_{22} f = 0\} \\
&= \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{21} f = X_{11} f = 0\} \cup \\
&\qquad \{x \in \mathbb{R}^4 : X_1 f = X_2 f = X_{21} f = X_{22} f = 0\} \\
&=: S_f^1 \cup S_f^3.
\end{aligned}
$$

By using the same technique used in the proof of Lemma 4.2, we can prove that there exists a semi-algebraic open dense set $D_d \subset \mathbb{R}_d[x]$ such that for each $f \in D_d$, each of the sets S_f^1, S_f^2, S_f^3 are finite. Then, we have the following:

Lemma 4.8. *There exists a semi-algebraic open dense set $D_d \subset \mathbb{R}_d[x]$ such that for every $f \in D_d$, the set $\Omega_f = \Omega_f^1 \cup \Omega_f^2$ is finite, so $\xi_f(x) \neq 0$ for $x \notin \Omega_f$.*

Let

$$
Gf := X_{11} f X_{22} f - X_{21} f X_{12} f. \tag{6}
$$

Then by (4), $\Gamma_f = \{X_1 f = X_2 f = Gf = 0\}$.

Now we are able to describe numerically the condition of tangency of Γ_f to a fiber of f.

Lemma 4.9. *Let $f \in I_d \cap D_d$. Let $x \in \Gamma_f - (\Omega_f \cup Cr(f))$ where $Cr(f)$ is still the set of critical points of f, which is finite. Then $[d_x(Gf)](\xi_f(x)) = 0$ if and only if Γ_f is tangent to the fiber $f^{-1}(f(x))$ at x.*

Proof. Note that $\xi_f(x) \neq 0$ and $\xi_f(x)$ is a tangent vector of $V_f \cap f^{-1}(f(x))$ at x.

Suppose that $[d_x(Gf)](\xi_f(x)) = 0$. Let us prove that $\xi_f(x)$ is a tangent vector of Γ_f at x. We have $[d_x(X_1 f, X_2 f, Gf)](\xi_f(x)) =$

$$
= \begin{pmatrix} X_{11} f & X_{21} f & X_{31} f & X_{41} f \\ X_{12} f & X_{22} f & X_{32} f & X_{42} f \\ X_1 G & X_2 G & X_3 G & X_4 G \end{pmatrix}_x \begin{pmatrix} -X_{21} f \\ X_{11} f \\ 0 \\ 0 \end{pmatrix}_x = \begin{pmatrix} 0 \\ Gf(x) \\ [d_x(Gf)](\xi_f(x)) \end{pmatrix}.
$$

Hence $[d_x(X_1 f, X_2 f, Gf)](\xi_f(x)) = 0$, which means that $\xi_f(x)$ is a tangent vector of Γ_f at x. Therefore, Γ_f is tangent to the fiber $f^{-1}(f(x))$ at x.

Now suppose that Γ_f is tangent to the fiber $f^{-1}(f(x))$ at x. Let η be a non-zero tangent vector of Γ_f at x, then $[d_x(Gf)](\eta) = 0$. Since $\Gamma_f \subset V_f$, then $\eta \in T_x V_f$. On the other hand, by the assumption, it follows that $\eta \in T_x f^{-1}(f(x))$. Consequently, $\eta \in T_x(V_f \cap f^{-1}(f(x)))$. Moreover, the two non-zero vectors η and $\xi_f(x)$ have to be linearly dependent. This implies that $[d_x(X_1 f, X_2 f, Gf)](\xi_f(x)) = 0$. □

For $f \in \mathbb{R}_d[x]$, we define the norm

$$||f|| := (\sum_i c_i^2)^{1/2}$$

where c_i is a coefficient of f and the sum is taken over the set of coefficients of f. Then the distance between $f, h \in \mathbb{R}_d[x]$ is defined by $||f - h||$.

Let $\pi : \mathbb{R}^4 \times \mathbb{R}_d[x] \to \mathbb{R}_d[x]$ be the projection map. Let $S := \Theta^{-1}(\{(0,0)\} \times \Sigma_2)$ where Θ is defined in Lemma 4.2 and Σ_2 is still the algebraic subset of M_2, constituted of degenerate matrices. Note that if $(x, f) \in S$, then $x \in \Gamma_f$. Let π_S be the restriction of π on S, then a fiber $(\pi_S)^{-1}(f)$ of π_S is just $\Gamma_f \times \{f\}$. By Theorem 2.5, there exists a semi-algebraic subset Z of codimension 1 of $\mathbb{R}_d[x]$ such that f is a semi-algebraic locally trivial fibration over each connected component of $T_d := \mathbb{R}_d[x] - Z$. It is clear that T_d contains a semi-algebraic open dense subset of $\mathbb{R}_d[x]$. Let $A_d \subset I_d \cap T_d$ such that A_d is semi-algebraic open dense. Now fix a polynomial $f \in A_d$. The set V_f is algebraic, smooth, $dimV_f = 2$ or V_f is empty, moreover Γ_f is algebraic, smooth, $dim\Gamma_f = 1$ or Γ_f is empty. Suppose that $\Gamma_f \neq \emptyset$.

For each $f \in A_d \cap D_d$, since Γ_f is an algebraic set, it has only a finite number of connected components $\Gamma_f^1, \ldots, \Gamma_f^s$, this number is bounded uniformly by $c(d)$ where $c(d)$ is the constant defined in Remark 4.4. Let $\kappa(f)$ be the number of connected components of Γ_f such that each of them is contained in a fiber of f (not necessarily the same) and let $\lambda(f)$ be the number of connected components of Γ_f which are not contained in any fiber of f. So $\lambda(f) + \kappa(f) \leq c(d)$. Now let us fix a polynomial $f \in A_d \cap D_d$. Suppose that $\kappa(f) > 0$. Denote $\epsilon := (\alpha, \beta, \gamma)$ and set

$$f_\epsilon := f + \alpha x_2 + \frac{\beta}{2}x_2^2 + \gamma x_4, \tag{7}$$

a perturbation of f by a form of degree 2. Let $|\epsilon| := \max\{|\alpha|, |\beta|, |\gamma|\}$ be the "size" of the perturbation. If ϵ is small enough, by the triviality given by Theorem 2.5, for each connected component θ of Γ_f, there exists a connected component θ_ϵ of Γ_{f_ϵ} which is close to θ. We call θ_ϵ the connected component **corresponding** to θ.

The following lemma is the key to prove the density in the theorem 4.1.

Lemma 4.10. *There exists $\epsilon = (\alpha, \beta, \gamma)$ such that $\lambda(f_\epsilon) > \lambda(f)$. Moreover*

$$\lambda(f_{t\epsilon}) = \lambda(f_{(t\alpha, t\beta, t\gamma)}) > \lambda(f)$$

for every $t \in (0, 1]$.

Proof. We write $\Gamma_f = \bigcup_{i=1}^{\lambda(f)} \Gamma_i \cup \bigcup_{i=\lambda(f)+1}^{\lambda(f)+\kappa(f)} \Gamma_i$, where

  (i) $\Gamma_f^1, \ldots, \Gamma_f^{\lambda(f)}$ are the connected components of Γ_f which are not contained in any fiber of f,
  (ii) $\Gamma_f^{\lambda(f)+1}, \ldots, \Gamma_f^{\lambda(f)+\kappa(f)}$ are the connected components of Γ_f which are contained in a fiber of f (not necessarily the same).

By Lemma 4.9, it follows that $\xi_f(x) \neq 0$ and $[d_x(Gf)](\xi_f(x)) = 0$ for every $x \in \bigcup_{i=\lambda(f)+1}^{\lambda(f)+\kappa(f)} \Gamma_i - (\Omega_f \cup Cr(f))$, recall that $\xi_f(x)$ is defined by (5).

Let us fix $\theta = \Gamma_f^{\lambda(f)+1}$, a connected component of Γ_f which is contained in a fiber of f. We look for a perturbation f_ϵ satisfying the following conditions:

(c1) The connected component θ_ϵ of Γ_{f_ϵ} corresponding to θ is not contained in any fiber of f_ϵ.

(c2) The connected components of Γ_{f_ϵ} corresponding to $\Gamma_f^1, \ldots, \Gamma_f^{\lambda(f)}$ are not contained in any fiber of f_ϵ neither.

Firstly we establish (c1). By Lemma 4.8, the set Ω_f is finite. Since $f \in J_d$, by Lemma 4.5, f is a Morse function, then $Cr(f)$ is finite. Consequently, $\theta - (\Omega_f \cup Cr(f)) \neq \emptyset$. Let us take a point $b = (b_1, b_2, b_3, b_4) \in \theta - (\Omega_f \cup Cr(f))$. Let $\epsilon = (\alpha, \beta, \gamma)$ be a solution of the following system of equations:

$$\left\{ \begin{array}{r} \alpha + b_2\beta + b_3\gamma = 0 \\ \beta + b_1\gamma = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \alpha = -b_2\beta - b_3\gamma = (b_1 b_2 - b_3)\gamma \\ \beta = -b_1\gamma \end{array} \right.$$

such that $\gamma \neq 0$. Note that by choosing γ small, we can make the size of ϵ arbitrary small. Recall that $X_3 = \dfrac{\partial}{\partial x_3}, X_4 = \dfrac{\partial}{\partial x_4}$. Now we have

$$\begin{array}{lll} X_i f_\epsilon & = & X_i f \qquad \text{for } i \in \{1, 3, 4\}, \\ X_2 f_\epsilon & = & X_2 f + \alpha + \beta x_2 + \gamma x_3, \\ X_{ij} f_\epsilon & = & X_{ij} f \qquad \text{for } (i,j) \in \{1,4\} \times \{1,4\}, (i,j) \neq (2,2), (3,2), \\ X_{22} f_\epsilon & = & X_{22} f + \beta + \gamma x_1. \end{array}$$

We remark the following properties.

Claim 4.11. *The perturbation f_ϵ fixes the point b in Γ_f, i.e., $b \in \Gamma_{f_\epsilon}$.*

Proof. We have

$$G f_\epsilon = X_{11} f_\epsilon X_{22} f_\epsilon - X_{21} f_\epsilon X_{12} f_\epsilon = X_{11} f (X_{22} f + \beta + \gamma x_1) - X_{21} f X_{12} f = G f + (\beta + \gamma x_1) X_{11} f.$$

By our choice of ϵ, it is easy to see that $X_1 f_\epsilon(b) = X_2 f_\epsilon(b) = G f_\epsilon(b) = 0$. This proves the claim. $\qquad \square$

Claim 4.12. *The connected component θ_ϵ of Γ_{f_ϵ} which contains b is not contained in any fiber of f_ϵ. In fact $\theta_\epsilon \pitchfork_b f_\epsilon^{-1}(f_\epsilon(b))$.*

Proof. First of all, we prove that $b \notin \Omega_{f_\epsilon} \cup Cr(f_\epsilon)$. Since $X_i f_\epsilon(b) = X_i f(b)$ for $i = 1, 2, 3, 4$, we deduce that $\nabla f_\epsilon(b) = \nabla f(b)$ where ∇f_ϵ and ∇f denote respectively the (Riemannian) gradient of f_ϵ and f. Since $b \notin Cr(f)$, b is not a (Riemannian) critical point of f and hence b is not a (Riemannian) critical point of f_ϵ. This implies that $f_\epsilon^{-1} f_\epsilon((b))$ is smooth at b. Moreover $T_b f_\epsilon^{-1} f_\epsilon((b)) = T_b f^{-1} f((b))$. On the other hand, since $X_{ij} f_\epsilon(b) = X_{ij} f(b)$ ($i = 1, 2, 3, 4, j = 1, 2$) by our choice of ϵ, it follows that $d_b(\nabla^h f) =$

$$\begin{pmatrix} X_{11} f & X_{21} f & X_{31} f & X_{41} f \\ X_{12} f & X_{22} f & X_{32} f & X_{42} f \end{pmatrix}_b = \begin{pmatrix} X_{11} f_\epsilon & X_{21} f_\epsilon & X_{31} f_\epsilon & X_{41} f_\epsilon \\ X_{12} f_\epsilon & X_{22} f_\epsilon & X_{32} f_\epsilon & X_{42} f_\epsilon \end{pmatrix}_b = d_b(\nabla^h f_\epsilon).$$

Hence $T_b V_f = Ker d_b(\nabla^h f) = Ker d_b(\nabla^h f_\epsilon) = T_b V_{f_\epsilon}$. Note that $X_{11} f_\epsilon(b) = X_{11} f(b) \neq 0$ and $X_{21} f_\epsilon(b) = X_{21} f(b) \neq 0$, consequently, $b \notin \Omega_{f_\epsilon}$. So $b \notin \Omega_{f_\epsilon} \cup Cr(f_\epsilon)$.

Now by Lemma 4.9, it is sufficient to show that $[d_b(G f_\epsilon)](\xi_{f_\epsilon}(b)) \neq 0$. Note that

$$G f_\epsilon = G f + (\beta + \gamma x_1) X_{11} f,$$

$$\xi_{f_\epsilon} = -(X_{21} f_\epsilon) X_1 + (X_{11} f_\epsilon) X_2 = -(X_{21} f) X_1 + (X_{11} f) X_2 = \xi_f$$

Hence

$$
\begin{aligned}
[d_b(Gf_\epsilon)](\xi_{f_\epsilon}(b)) &= [d_b(Gf + (\beta + \gamma x_1)X_{11}f)](\xi_f(b)) \\
&= [d_b(Gf)](\xi_f(b)) + [d_b\{(\beta + \gamma x_1)X_{11}f\}](\xi_f(b)) \\
&= [d_b(Gf)](\xi_f(b)) + [(\beta + \gamma x_1)d(X_{11}f) \\
&\quad + (X_{11}f)d\{(\beta + \gamma x_1)\}]_b(\xi_f(b)) \\
&= [d_b(Gf)](\xi_f(b)) + (\beta + \gamma b_1)[d_b(X_{11}f)](\xi_f(b)) \\
&\quad + (X_{11}f(b))[d_b(\beta + \gamma x_1)](\xi_f(b)). \\
&= [d_b(Gf)](\xi_f(b)) + (\beta + \gamma b_1)[d_b(X_{11}f)](\xi_f(b)) \\
&\quad - \gamma[X_{11}f(b)].[X_{21}f(b)].
\end{aligned}
\tag{8}
$$

Since $[d_b(Gf)](\xi_f(b)) = 0$ by our initial setting and $(\beta + \gamma b_1) = 0$ by our choice of ϵ, it follows from Lemma 4.8 that $[d_b(Gf_\epsilon)](\xi_{f_\epsilon}(b)) = -\gamma[X_{11}f(b)].[X_{21}f(b)] \neq 0$ for $\epsilon \neq 0$. This ends the proof of the claim. $\qquad\square$

So with Claim 4.12, we have established (c1). Now we try to establish (c2). To do so, the following claim is sufficient.

Claim 4.13. *For each $i = 1, \ldots, \lambda(f)$, let $\Gamma^i_{f_\epsilon}$ be the connected component of Γ_{f_ϵ} corresponding to Γ^i_f. There exists a constant $0 < \delta$ such that if $|\epsilon| \leq \delta$, then $\Gamma^i_{f_\epsilon}$ is not contained in any fiber of f_ϵ.*

Proof. Since Γ^i_f is not contained in any fiber of f, it follows that Γ^i_f is almost everywhere transverse to the fibers of f. By Lemma 4.8, the set $\Omega_f \cup Cr(f)$ is finite. Hence, we can choose a point $a_i \in \Gamma^i_f - (\Omega_f \cup Cr(f))$ such that $\Gamma^i_f \pitchfork_{a_i} f^{-1}(f(a_i))$. By Lemma 4.9, $[d_{a_i}(Gf)](\xi_f(a_i)) \neq 0$. Let

$$
m_1 := \min_{i=1,\ldots,s_1} |[d_{a_i}(Gf)](\xi_f(a_i))| > 0.
$$

Let $\zeta > 0$ such that $B(a_i, \zeta) \cap \{|[d(Gf)](\xi_f)| \leq \frac{3m_1}{4}\} = \emptyset$ where $B(a_i, \zeta) := \{x \in \mathbb{R}^4 : \|x - a_i\| \leq \zeta\}$. Set

$$
m_2 := \max_{i=1,\ldots,s_1; x \in B(a_i,\zeta)} |[d_x(X_{11}f)](\xi_f(x))| < +\infty,
$$

$$
m_3 := \max_{i=1,\ldots,s_1; x \in B(a_i,\zeta)} |X_{11}f(b)|.|X_{21}f(b)| < +\infty,
$$

$$
m_4 := \max_{i=1,\ldots,s_1; x \in B(a_i,\zeta)} \|x - b\| < +\infty.
$$

Note that $m_3, m_4 > 0$. Let $r > 0$ such that

$$
r \leq \begin{cases} \min\{\dfrac{m_1}{4m_2 m_4}, \dfrac{m_1}{4m_3}\} & \text{if } m_2 \neq 0 \\[2mm] \dfrac{m_1}{4m_3} & \text{if } m_2 = 0. \end{cases}
$$

and such that for $|\epsilon| < r$, $f_\epsilon \in A_d \cap D_d$ and the intersection $B(a_i, \zeta) \cap \Gamma^i_{f_\epsilon}$ is a curve. Since the set $\Omega_{f_\epsilon} \cup Cr(f_\epsilon)$ is finite, we can choose a point $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3}, \sigma_{i4}) \in B(a_i, \zeta) \cap \Gamma^i_{f_\epsilon}$ such that $\sigma_i \notin \Omega_{f_\epsilon} \cup Cr(f_\epsilon)$. Let us prove that $[d_{\sigma_i}(Gf_\epsilon)](\xi_{f_\epsilon}(\sigma_i)) \neq 0$. In fact, by the same computation as (8), we have

$$
\begin{aligned}
[d_{\sigma_i}(Gf_\epsilon)](\xi_{f_\epsilon}(\sigma_i)) &= [d_{\sigma_i}(Gf)](\xi_f(\sigma_i)) + (\beta + \gamma\sigma_{i1})[d_{\sigma_i}(X_{11}f)](\xi_f(\sigma_i)) \\
&\quad - \gamma[X_{11}f(\sigma_i)].[X_{21}f(\sigma_i)].
\end{aligned}
$$

So

$$
\begin{aligned}
|[d_{\sigma_i}(Gf_\epsilon)](\xi_{f_\epsilon}(\sigma_i))| &\geq |[d_{\sigma_i}(Gf)](\xi_f(\sigma_i))| - |(\beta + \gamma\sigma_{i1})|.|[d_{\sigma_i}(X_{11}f)](\xi_f(\sigma_i))| \\
&\quad - |\gamma|.|X_{11}f(\sigma_i)|.|X_{21}f(\sigma_i)|.
\end{aligned}
$$

We note the following facts:

i) Since $\sigma_i \in B(a_i, \zeta)$ and $B(a_i, \zeta) \cap \{|[d\sigma_i(Gf)](\xi_f)| \leq \frac{3m_1}{4}\} = \emptyset$, it follows that

$$|[d_{\sigma_i}(Gf)](\xi_f(\sigma_i))| > \frac{3m_1}{4}.$$

ii) $|(\beta + \gamma\sigma_{i1})|.|[d_{\sigma_i}(X_{11}f)](\xi_f(\sigma_i))| = |(-b_1 + \sigma_{i2})\gamma|.|[d_{\sigma_i}(X_{11}f)](\xi_f(\sigma_i))| \leq m_4 r m_2 \leq \frac{m_1}{4}$.

iii) $|\gamma|.|X_{11}f(\sigma_i)|.|X_{21}f(\sigma_i)| \leq r m_3 \leq \frac{m_1}{4}$.

These imply that

$$|[d_{\sigma_i}(Gf_\epsilon)](\xi_{f_\epsilon}(\sigma_i))| \geq \frac{3m_1}{4} - \frac{m_1}{4} - \frac{m_1}{4} = \frac{m_1}{4} > 0.$$

Then by Lemma 4.9, we deduce that $\Gamma^i_{f_\epsilon} \pitchfork_{\sigma_i} f^{-1}(f_\epsilon(\sigma_i))$. This ends the proof of the claim.

$\square$

By construction, $\lambda(f_\epsilon) \geq \lambda(f)+1$. Moreover, $\lambda(f_{t\epsilon}) \geq \lambda(f)+1$ for every $t \in (0,1]$. Hence, the lemma is proved. $\square$

Lemma 4.10 shows that, after a small appropriate perturbation f_ϵ, the number of connected components of Γ_{f_ϵ}, which are not contained in any fiber of f_ϵ, increases. So we obtain an important corollary as follows.

Corollary 4.14. *The intersection $M_d \cap A_d \cap D_d$, where M_d is defined by (3), is dense in $\mathbb{R}_d[x]$.*

Proof. Since the number $\lambda(f)$ is bounded from above by $c(d)$, repeated applications of Lemma 4.10 give a polynomial $h \in M_d$ which can be chosen arbitrarily close to f. The corollary follows. $\square$

The following lemma proves the semi-algebraicity of the set $M_d \cap A_d \cap D_d$.

Lemma 4.15. *The set $M_d \cap A_d \cap D_d$, where M_d is defined by (3), is a semi-algebraic set.*

Proof. For $f \in A_d \cap D_d$, denote $\Gamma^1_f, \ldots, \Gamma^{k_f}_f$ be the connected components of Γ_f which are semi-algebraic sets. Then

$$M_d \cap A_d \cap D_d = \{f \in A_d \cap D_d : \quad \exists \sigma_i \in \Gamma^i_f - (\Omega_f \cup Cr(f)) :$$
$$[d_{\sigma_i}(Gf)](\xi_f(\sigma_i)) \neq 0, i = 1, \ldots, k_f\}.$$

This set is clearly a semi-algebraic set by Tarski-Seidenberg principle [4, 5, 7] and Theorem 2.5. $\square$

Now we have all necessary ingredients to prove the theorem 4.1.

Proof of Theorem 4.1. By Corollary 4.14 and lemma 4.15, it follows that $M_d \cap A_d \cap D_d$ is semi-algebraic and dense in $\mathbb{R}_d[x]$. Then by Theorem 2.4, $M_d \cap A_d \cap D_d$ contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$. Consequently, M_d contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$. This completes the proof of the theorem 4.1. $\square$

Form Theorem 4.1, we obtain the following corollary.

Corollary 4.16. *Generically, the set V_f does not contain a horizontal curve, i.e., the set*

$$P_d = \{f \in \mathbb{R}_d[x] : V_f \text{ does not contain a horizontal curve}\}$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

Proof. Let $f \in B_d \cap M_d$. By the proof of Lemma 4.2, it follows that $B_d \subset K_d$ where K_d is defined in Theorem 3.1, so V_f is smooth of dimension 2 or is empty. Assume that $V_f \neq \emptyset$. Let us prove that $f \in P_d$.

By contradiction, suppose that $\gamma \subset V_f$ is a connected horizontal curve, let $\gamma(t)$ be a parametrization of γ. Since $\gamma(t)$ is almost everywhere differentiable and $\gamma \subset V_f$, we have $\dot{\gamma}(t) \in T_{\gamma(t)}V_f$ for almost t. On the other hand, since γ is horizontal, it follows that $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$. Hence, for almost t, we have $\dot{\gamma}(t) \in T_{\gamma(t)}V_f \cap \Delta_{\gamma(t)}$, so $T_{\gamma(t)}V_f \cap \Delta_{\gamma(t)} \neq \{0\}$. This implies that $\gamma(t) \in \Gamma_f$ for almost t. By the absolute continuousness of $\gamma(t)$ and the smoothness of Γ_f from Lemma 4.2, we have $\gamma \subset \Gamma_f$. On the other side, since $\Delta_{\gamma(t)} \subset Ker(d_{\gamma(t)}f)$, we get $\dfrac{d}{dt}[f(\gamma(t))] = [d_{\gamma(t)}f](\dot{\gamma}(t)) = 0$ for almost t. Hence $f(\gamma(t)) = c$ where c is a constant, which implies that γ is contained in the fiber $f^{-1}(c)$. Consequently, Γ_f has a connected component which is contained in a fiber. This contradicts Theorem 4.1. The corollary is proved. $\square$

Remark 4.17. Let $f \in B_d \cap M_d$, if f is non constant on each connected component of Γ_f, then V_f, and hence Γ_f, do not contain a horizontal curve.

5. Trajectories of horizontal gradient

In this section, we prove that for a generic polynomial f, every trajectory of $\nabla^h f$ in a given compact set B always has a limit. Precisely, we will prove the following.

Theorem 5.1. *Generically, the trajectories of $\nabla^h f$ in B have a limit, i.e., the set*

$$N_d = \{f \in \mathbb{R}_d[x] : \text{ the trajectories of } \nabla^h f \text{ have a limit on } \partial B \cup V_f\} \qquad (9)$$

contains a semi-algebraic open dense set in $\mathbb{R}_d[x]$.

Let $d_{V_f}(.)$ be the distance function to V_f with respect to the usual Euclidean metric on $\mathbb{R}^4$. By Łojasiewicz inequality ([18]), there exist some constants $C_1, C_2 > 0, \gamma \geq 1 \geq \beta > 0$ such that for all $x \in B$, we have

$$C_1 d_{V_f}^{\gamma}(x) \leq ||\nabla^h f(x)|| \leq C_2 d_{V_f}^{\beta}(x). \qquad (10)$$

We have the following lemma.

Lemma 5.2. *For $f \in L_d$, where L_d is defined in Proposition 3.2, we can choose the exponents $\gamma = \beta = 1$ in the Łojasiewicz inequality (10), i.e., there exist some constants $0 < C_1 \leq C_2$ such that for all $x \in B$, we have*

$$C_1 d_{V_f}(x) \leq ||\nabla^h f(x)|| \leq C_2 d_{V_f}(x). \qquad (11)$$

Proof. The idea of proof is similar to Lemma 5.15 in [8]. By Proposition 3.2, for $f \in L_d$, $\nabla^h f$ is a submersion on V_f. This implies that for each $x \in V_f$, the Jacobian matrix of $\nabla^h f : \mathbb{R}^4 \to \mathbb{R}^2$ is of constant rank 2. By constant rank theorem, for all $x \in V_f$, there exist a diffeomorphism $u : U_1^x \to U^x$ from a neighborhood U_1^x of $0 \in \mathbb{R}^4$ on a neighborhood U^x of x, $u(0) = x$, and a diffeomorphism $w : W \to W_1$ from a neighborhood W of $0 \in \mathbb{R}^2$ on a neighborhood W_1 of $0 \in \mathbb{R}^2$, $w(0) = 0$ such that

$$A(y_1, y_2, y_3, y_4) = (y_1, y_2)$$

where $A = w \circ \nabla^h f \circ u$. It is clear that $u^{-1}(V_f \cap U^x) = \{y_1 = y_2 = 0\} \cap U_1^x$ and that

$$||A(y)|| = ||(y_1, y_2)|| = dist(y, V_1 f) \tag{12}$$

where $dist(., V_1 f)$ is the distance to $V_1 f$ with respect to the standard metric on $\mathbb{R}^4$. Since u and w are diffeomorphism (defined on the convex sets), they are bi-Lipschitz, i.e., u, v and u^{-1}, v^{-1} are Lipschitz. Hence, by (12), there exist some constants $0 < C_1^x \leq C_2^x < +\infty$ such that for all $z \in U^x$, we have

$$C_1^x d_{V_f}(z) \leq ||\nabla^h f(z)|| \leq C_2^x d_{V_f}(z). \tag{13}$$

By compactness of $B \cap V_f$, we can find a finite cover of $B \cap V_f$ by some open sets U^{x_i}, $i = 1, \ldots, k$, such that (13) holds in each U^{x_i}. Note that there exists $r > 0$ such that for all $z \in \tilde{B} = B - \bigcup_i U^{x_i}$, we have $d_{V_f}(z) \geq r$, so let

$$\tilde{C}_1 = \min_{\tilde{B}} \frac{||\nabla^h f||}{d_{V_f}}, \tilde{C}_2 = \max_{\tilde{B}} \frac{||\nabla^h f||}{d_{V_f}}.$$

Then

$$\tilde{C}_1 d_{V_f}(z) \leq ||\nabla^h f(z)|| \leq \tilde{C}_2 d_{V_f}(z)$$

for all $z \in \tilde{B}$. Finally, let

$$C_1 = \min\{C_1^{x_1}, \ldots, C_1^{x_k}, \tilde{C}_1\}, C_2 = \max\{C_2^{x_1}, \ldots, C_2^{x_k}, \tilde{C}_2\}.$$

Then for all $z \in B$, we have

$$C_1 d_{V_f}(z) \leq ||\nabla^h f(z)|| \leq C_2 d_{V_f}(z).$$

The lemma is proved.   □

Recall that $g_{SR} = dx_1^2 + dx_2^2$ is a sub-Riemannian metric on Δ. Let $\delta_{SR} = d_{V_f}^2 g_{SR}$. Note that the metrics δ_{SR} depends on the polynomial f. Moreover, it is degenerate on V_f.

Let $<< ., . >>$ and $||.||$ denote respectively the scalar product and the norm for the metric g_{SR}, let $< ., . >$ and $|.|$ denote respectively the scalar product and the norm for the metric δ_{SR}. For a horizontal curve $\alpha : (t_1, t_2) \rightarrow \mathbb{R}^4$, its length with respect to g_{SR} and δ_{SR}, denoted respectively by $l_g(\alpha(t))$ and $l_\delta(\alpha(t))$ are given by the following formulas.

$$l_g(\alpha(t)) = \int_0^T ||\dot{\alpha}(t)||_{\alpha(t)} dt, \tag{14}$$

$$\begin{aligned} l_\delta(\alpha(t)) &= \int_0^T |\dot{\alpha}(t)|_{\alpha(t)} dt \\ &= \int_0^T d_{V_f}(\alpha(t))||\dot{\alpha}(t)||_{\alpha(t)} dt. \end{aligned} \tag{15}$$

The horizontal gradient for the metric δ_{SR} can be computed from the one for the metric g_{SR}, this is given by the following lemma.

Lemma 5.3. *Let* $^\delta\nabla f = (u_1, u_2, u_3, u_4)$ *and* $^\delta\nabla^h f = (u_1, u_2)$, *be respectively the gradient and the horizontal gradient of f with respect to the metric $\delta := \delta_{SR} + \omega_3^2 + \omega_4^2$ where $\omega_3 = dx_3 - x_1 dx_2$, $\omega_4 = dx_4 - x_3 dx_2$. Then, in the coordinate system $\{X_1, X_2, X_3, X_4\}$, we have*

$$\begin{cases} u_i = \dfrac{X_i f}{d_{V_f}^2}, (i = 1, 2) \\ u_i = X_i f, (i = 3, 4) \end{cases}$$

Which mean ${}^\delta\nabla^h f = \dfrac{\nabla^h f}{d_{V_f}^2}$ and $|{}^\delta\nabla^h f| = \dfrac{||\nabla^h f||}{d_{V_f}}$.

Proof. The proof is similar to the proof of Lemma 5.8 in [8]. □

The following lemma is similar to Theorem 5.10 in [8].

Lemma 5.4. *Let* $\alpha : (t_1, t_2) \to B$ *is a trajectory of* $\nabla^h f$. *Then for the metric* δ, *the length of* α *is bounded by* $\dfrac{|f(\alpha(t_2)) - f(\alpha(t_1))|}{C_1}$ *where* C_1 *is the constant (depended on B) in the Łojasiewicz inequality (11).*

Proof. The proof is obtained by the same way as the proof of Theorem 5.10 in [8]. □

The following property is similar to Proposition 5.1 in [8].

Proposition 5.5. *If* $x(t)$ *is not a constant trajectory of* $\nabla^h f$, *then* $\dfrac{d}{dt}(f(x(t))) > 0$ *which means that* f *is non constant on* $x(t)$. *So the there are no closed trajectories of* $\nabla^h f$.

Proof. The proof is analogue to the proof of Proposition 5.1 in [8]. □

Now for $a \in V_f - \Gamma_f$, let U be an open neighborhood of a such that $U \cap \Gamma_f = \emptyset$, let Y, Z be some open neighborhoods of $0 \in \mathbb{R}^2$, let

$$\varphi : \quad \begin{array}{ccc} Y & \to & V_f \cap U \\ (y_1, y_2) & \mapsto & \varphi(y_1, y_2) \end{array}$$

be a smooth local parameterization of V_f at a such that $\varphi(0) = a$. Set

$$\Phi : \quad \begin{array}{ccc} Y \times \mathbb{R}^2 & \to & U \\ y = (y_1, y_2, y_3, y_4) & \mapsto & \varphi(y_1, y_2) + y_3 X_1(\varphi(y_1, y_2)) + y_4 X_2(\varphi(y_1, y_2)) \end{array}$$

We have the following lemma.

Lemma 5.6. *The map* Φ *is a local diffeomorphism at* $0 \in \mathbb{R}^4$.

Proof. Note that $(d\Phi)_0(\mathbb{R}^2 \times \{0\}) = \{(d\varphi)_0(\mathbb{R}^2)\} \times \{0\} = T_a V_f \times \{0\}$ and $(d\Phi)_0(\{0\} \times \mathbb{R}^2) = \{0\} \times \Delta_a$. On the other side, $T_a V_f \pitchfork \Delta_a$ since $a \notin \Gamma_f$. Hence $(d\Phi)_0$ is surjective which implies that it is also bijective. The lemma follows. □

Remark 5.7. Let $dist$ and $dist_E$ be respectively the Riemannian distance with respect to the metric $g = dx_1^2 + dx_2^2 + \omega_3^2 + \omega_4^2$ and the usual Euclidean metric g_E on $\mathbb{R}^4$. From the property that all scalar products on a finite dimensional vector space are equivalent, it follows that $dist$ and $dist_E$ are equivalent on any compact subset of $\mathbb{R}^4$.

Let $0 \in W \subset Y \times Z$ such that W is compact and that $\Phi : W \to \Phi(W)$ is a diffeomorphism. Set

$$\tilde{V}_f := \Phi^{-1}(V_f) \subset \mathbb{R}^2 \times \{0\}.$$

Then it is easy to check that

$$
\begin{aligned}
d_{V_f}(.)|_{\Phi(W)} &= dist_E(., V_f)|_{\Phi(W)} \\
&\sim dist_E(., \tilde{V}_f)|_W \\
&= dist_E(., \{y_3 = y_4 = 0\})|_W \\
&=: d_{\tilde{V}_f}(.)|_W.
\end{aligned}
$$

Let $b \in \Phi(W)$ and set $\tilde{b} = \Phi^{-1}(b)$. Now set $B(\tilde{b}, R) := \{y \in \mathbb{R}^4 : |y_i - \tilde{b}_i| \leq R\} \subset W$, then $B_\Phi(b, R) := \Phi(B(\tilde{b}, R))$ is a closed neighborhood of b.

The following lemma is the key to prove the existence of the limits of the trajectories of $\nabla^h f$.

Lemma 5.8. *Let $f \in M_d \cap A_d \cap D_d \cap L_d$, we have the following.*

(I) *Let $\alpha : [t_1, t_2] \to B_\Phi(b, R)$ be a horizontal curve such that $\alpha(t_1) = b$ and $\alpha(t_2) \in \partial B_\Phi(b, R)$. Then $l_\delta(\alpha(t)) > C_{b,R} > 0$ where $C_{b,R}$ does not depend on α.*

(II) *For each $p \in \partial B_\Phi(a, \dfrac{R}{2})$ and for any horizontal curve $\alpha : [t_1, t_2] \to B_\Phi(a, R) - \mathring{B}_\Phi(a, \dfrac{R}{2})$ such that $\alpha(t_1) = p$ and $\alpha(t_2) \in \partial B_\Phi(a, R)$, we have $l_\delta(\alpha(t)) > C'_{a,R} > 0$ where $C'_{a,R}$ does not depend on α.*

Proof. (I) By compactness, it is sufficient to prove that $l_\delta(\alpha(t)) > 0$ for such $\alpha(t)$. Set $q := \alpha(t_2)$ and $\tilde{q} := \Phi^{-1}(q) = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)$. We have four cases to consider:

(i) $b \notin V_f$;

(ii) $b \in V_f$ and $|\tilde{q}_3 - \tilde{b}_3| = R$ or $|\tilde{q}_4 - \tilde{b}_4| = R$;

(iii) $b \in V_f$, $|\tilde{q}_3 - \tilde{b}_3| < R$ and $|\tilde{q}_4 - \tilde{b}_4| < R$.

Claim 5.9. *We have $l_g(\alpha(t)) \geq cR$ where c is a positive constant.*

Proof. By Lemma 5.6, Φ is a diffeomorphism, hence it is bi-Lipschitz, so by Remark 5.7, we get

$$l_g(\alpha(t)) \geq dist(b, q) \sim dist_E(b, q) \sim dist_E(\tilde{b}, \tilde{q}) \geq R.$$

$\square$

We need first the following claim whose proof follows easily from Claim 5.9 and the formulas (14), (15).

Claim 5.10. *If $d_{V_f}(\alpha(t)) \geq r > 0$, then $l_\delta(\alpha(t)) \geq r l_g(\alpha(t)) \geq crR$.*

Now consider case (i). Set $2r := d_{V_f}(b) > 0$. If $d_{V_f}(\alpha(t)) > r$ for all $t \in [t_1, t_2]$, then by Claim 5.10, it follows that

$$l_\delta(\alpha(t)) \geq crR > 0.$$

Otherwise, there exists $t_0 \in (t_1, t_2]$ such that $d_{V_f}(\alpha(t_0)) = r$ and $d_{V_f}(\alpha(t)) > r$ for $t \in [t_1, t_0]$. Then

$$
\begin{aligned}
l_\delta(\alpha(t)) &\geq l_\delta(\alpha(t)|_{[t_1,t_0]}) \\
&\geq r l_g(\alpha(t)|_{[t_1,t_0]}) \\
&\geq r\, dist(b, \alpha(t_0)) \\
&\sim r\, dist_E(b, \alpha(t_0)) \\
&\geq r^2 > 0
\end{aligned}
$$

Next, for case (ii), by the assumptions, we have

$$dist_E(\tilde{q}, \{y_3 = y_4 = 0\}) \geq R.$$

Hence

$$R \leq dist_E(\tilde{q}, \tilde{V}_f) \sim dist_E(q, V_f) = d_{V_f}(q)$$

which means that there is $r' > 0$ such that $d_{V_f}(q) \geq 2r'$. By applying the arguments as in the case (i), it follows that $l_\delta(\alpha(t)) > 0$.

Let us consider the case (iii), since $q \in \partial B_\Phi(b, R)$, we have $\tilde{q} \in \partial B(\tilde{b}, R)$. So $|\tilde{q}_1 - \tilde{b}_1| = R$ or $|\tilde{q}_2 - \tilde{b}_2| = R$. Let $\tilde{\alpha}(t) = \Phi^{-1}(\alpha(t))$. For $y \in W$, set

$$\tilde{\Delta}_y = d\Phi^{-1}(\Delta_{\Phi(y)}),$$

which defines a distribution $\tilde{\Delta}$ on W. Let $y \in W$ and $X, Y \in T_y\tilde{\Delta}_y$, the "pullbacks" $\tilde{g}_{SR}$ of g_{SR} and $\tilde{\delta}_{SR}$ of δ_{SR} on $\tilde{\Delta}$ at y are given respectively by

$$\tilde{g}_{SR}(X, Y) := g_{SR}(d_y\Phi(X), d_y\Phi(Y)),$$

$$\tilde{\delta}_{SR}(X, Y) := \delta_{SR}(d_y\Phi(X), d_y\Phi(Y)) = d_{V_f}^2(\phi(y)).g_{SR}(d_y\Phi(X), d_y\Phi(Y)).$$

Since all metrics on $\mathbb{R}^2$ are locally equivalent, by compactness, we have $\tilde{g}_{SR}(X, Y) \sim g_E|_{\tilde{\Delta}}$. Moreover $d_{V_f} \sim d_{\tilde{V}_f}$. So

$$\tilde{\delta}_{SR}(X, Y) \sim d_{\tilde{V}_f}^2.g_E|_{\tilde{\Delta}}.$$

Denote by $l_{\tilde{\delta}}(\tilde{\alpha}(t))$ the length of $\tilde{\alpha}(t)$ with respect to the metric $\tilde{\delta}_{SR}$. Then by this observation, it is sufficient to prove that $l_{\tilde{\delta}}(\tilde{\alpha}(t)) > 0$.

Without loss of generality, we may assume that $\tilde{\alpha}(t)$ is parametrized by its arc length with respect to the metric g_E, i.e., $\|\dot{\tilde{\alpha}}(t)\|_E = 1$ for $t \in [t_0, t_2]$, where $\|.\|_E$ denotes the Euclidean norm. Let $\xi(t)$ be the orthogonal projection of $\tilde{\alpha}(t)$ on $\tilde{V}_f$ with respect to the Euclidean metric g_E, so $\dot{\xi}(t)$ is the projection of $\dot{\tilde{\alpha}}(t)$ on $T_{\xi(t)}\tilde{V}_f = \mathbb{R}^2 \times \{0\}$. Note that $dist_E(\xi(t_1), \xi(t_2)) = \text{dist}_E(\tilde{b}, (\tilde{q}_1, \tilde{q}_2, 0, 0)) \geq R$. Denote by $\mathbb{G}(2, 4)$ the Grassmannian space of $2-$planes in $\mathbb{R}^4$, the distance between two planes $P, Q \in \mathbb{G}(2, 4)$ is given by

$$\rho(P, Q) := \max_{p \in P, \|p\|=1} \min_{q \in Q} \|p - q\|_E.$$

Since the map $\tilde{\Delta} : \mathbb{R}^4 \to \mathbb{G}(2, 4), y \mapsto \tilde{\Delta}_y$ is smooth, it is locally Lipschitz, so we may suppose that for all $x, y \in W$, we have

$$\rho(\tilde{\Delta}_x, \tilde{\Delta}_y) \leq K\|x - y\|_E,$$

where K is a positive constant. Hence for $t \in [t_0, t_2]$,

$$\rho(\tilde{\Delta}_{\alpha(t)}, \tilde{\Delta}_{\xi(t)}) \leq K\|\alpha(t) - \xi(t)\|_E.$$

Let $\dot{\eta}(t)$ be the orthogonal projection of $\dot{\tilde{\alpha}}(t)$ on $\tilde{\Delta}_{\xi(t)} \subset \{0\} \times \mathbb{R}^2$ with respect to the Euclidean metric g_E. Since $\tilde{\Delta}_{\xi(t)} \perp T_{\xi(t)}\tilde{V}_f$, we have $\dot{\tilde{\alpha}}(t) = \dot{\xi}(t) + \dot{\eta}(t)$. So

$$\|\dot{\xi}(t)\|_E = \|\dot{\tilde{\alpha}}(t) - \dot{\eta}(t)\|_E \leq \rho(\tilde{\Delta}_{\tilde{\alpha}(t)}, \tilde{\Delta}_{\xi(t)}) \leq K\|\tilde{\alpha}(t) - \xi(t)\|_E \leq Kd_{\tilde{V}_f}(\tilde{\alpha}(t)).$$

Consequently

$$
\begin{aligned}
l_\delta(\tilde{\alpha}) &= \int_{t_0}^{t_2} |\dot{\tilde{\alpha}}(t)| dt \\
&= \int_{t_0}^{t_2} d_{\tilde{V}_f}(\tilde{\alpha}(t)) dt \\
&\geq \frac{1}{K} \int_{t_0}^{t_2} ||\dot{\xi}(t)||_E dt \\
&\geq \frac{1}{K} dist_E(\xi(t_1), \xi(t_2)) \geq R.
\end{aligned}
$$

This end the proof of (I). It is clear that (II) follows easily from (I). The lemma follows. $\square$

Now we have all the ingredients necessary to prove Theorem 5.1.

Proof of Theorem 5.1. Let $f \in M_d \cap A_d \cap D_d \cap L_d$ and let $\alpha(t) \subset B$ be a trajectory of f in B, assume that $\alpha(t)$ is parametrized by the values of f. If $\alpha(t)$ does not approach to V_f, then by the same arguments as Proposition 5.4 in [8], it follows that $\alpha(t)$ has a limit on ∂B. So from now on, assume that $\alpha(t)$ approaches to V_f, i.e., there exists a sequence $t_m \to t_0 < +\infty$ such that

$$d_{V_f}(\alpha(t_m)) \to 0.$$

Since $\alpha(t)$ is parametrized by the values of f, we have

$$dist_E(\alpha(t), f^{-1}(t_0) \cap B) \to 0 \text{ when } t \to t_0,$$

where $dist_E$ denotes the distance function to V_f with respect to the usual Euclidean metric on $\mathbb{R}^4$. Thus

$$dist_E(\alpha(t), V_f \cap f^{-1}(t_0) \cap B) \to 0 \text{ when } t \to t_0. \tag{16}$$

We need to prove that $\alpha(t)$ has only one accumulation point in $V_f \cap f^{-1}(t_0) \cap B$. By contradiction, assume that $\alpha(t)$ has two accumulation points a, a' in $V_f \cap f^{-1}(t_0) \cap B$.

If $a \notin \Gamma_f$, then there are a local diffeomorphism Φ and a "box" $B_\Phi(a, R)$ satisfying the conclusions such that Lemma 5.8 holds. Then it is clear that for R small enough, $\alpha(t) \cap \left(B_\Phi(a, R) - \mathring{B}_\Phi(a, \frac{R}{2}) \right)$ contains infinity many components with one end point in $\partial B_\Phi(a, R)$ and another in $\partial B_\Phi(a, \frac{R}{2})$. According to Lemma 5.8, the length of these components with respect to the metric δ_{SR} is bounded from below by a positive constant, which implies that $l_\delta(\alpha(t)) = +\infty$. This contradicts Lemma 5.4.

If $a' \in \Gamma_f$, by the same arguments, we get a contradiction. So suppose that $a, a' \in \Gamma_f$. By Theorem 4.1, the intersection $f^{-1}(t_0) \cap \Gamma_f$ is finite, so let R be small enough such that $B_\Phi(a, R) \cap f^{-1}(t_0) \cap \Gamma_f = \{a\}$. Since $\alpha(t)$ accumulates to a, it intersects $\partial B_\Phi(a, R)$ infinitely many times. By compactness, the set $\partial B_\Phi(a, R) \cap \alpha(t)$ has an accumulation point, denoted by a_1. By (16), it is clear that $a_1 \in f^{-1}(t_0)$. Hence $a_1 \notin \Gamma_f$. By the same arguments as above, we get again a contradiction. So we have proved $f \in N_d$, where N_d is defined by (9). This ends the proof of Theorem 5.1. $\square$

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